Economic application of a second-order differential equation

If the instant price adjustment in a given market is governed by the following model:

$$\begin{cases} D = 300 - 2p + 4p' - p'' \\ O = 50 + 3p \\ p'(t) = 0.2(D - O) \end{cases}$$

1. Formulate the resulting differential equation from the model and find the general solution.
2. Given the initial conditions $p(0) = 12$ and $p'(0) = 1$, find the particular solution and analyze if it is a dynamically stable equilibrium, justifying your answer.

Solution

1. Combining the 3 equations:

$$p' = 0.2(300 - 2p + 4p' - p'' - 50 - 3p)$$

$$p' = 0.2(250 - p + 4p' - p'')$$

$$p' = 50 - 0.2p + 0.8p' - 0.2p''$$

$$0.2p' = 50 - 0.2p - 0.2p''$$

Dividing everything by 0.2:

$$p' = 250 - p - p''$$

$$p'' + p' + 5p = 250$$

2. Finding the homogeneous solution:

$$r^2 + r + 5 = 0$$

The roots are:

$$r_1 = -\frac{1}{2} + i\frac{\sqrt{19}}{2}$$

$$r_2 = -\frac{1}{2} - i\frac{\sqrt{19}}{2}$$

Therefore, the homogeneous solution is:

$$p_H = e^{-\frac{1}{2}t}\left[C_1 \sin\left(\frac{\sqrt{19}}{2}t\right) + C_2 \cos\left(\frac{\sqrt{19}}{2}t\right)\right]$$

Finding the particular solution:

$$p_c = K$$

$$p_c' = p_c'' = 0$$

$$0 + 0 + 5K = 250$$

Thus, $K = 50$:

$$p_c = 50$$

The general solution is:

$$p_g = p_H + p_c = e^{-\frac{1}{2}t}\left[C_1 \sin\left(\frac{\sqrt{19}}{2}t\right) + C_2 \cos\left(\frac{\sqrt{19}}{2}t\right)\right] + 50$$

Taking into account the initial conditions:

$$[C_2] + 50 = 12$$

From this we get $C_2 = -38$
On the other hand, we need to derive p:

$$p = e^{-\frac{1}{2}t}C_1 \sin\left(\frac{\sqrt{19}}{2}t\right) + e^{-\frac{1}{2}t}C_2 \cos\left(\frac{\sqrt{19}}{2}t\right) + 50$$

$$p' = \frac{-1}{2}e^{-t/2}C_1 \sin\left(\frac{\sqrt{19}}{2}t\right) + e^{-\frac{1}{2}t}C_1 \cos\left(\frac{\sqrt{19}}{2}t\right)\frac{\sqrt{19}}{2} + \frac{-1}{2}e^{-t/2}C_2 \cos\left(\frac{\sqrt{19}}{2}t\right) - e^{-t/2}C_2 \sin\left(\frac{\sqrt{19}}{2}t\right)\frac{\sqrt{19}}{2}$$

Evaluating at the point:

$$p' = C_1\frac{\sqrt{19}}{2} + \frac{-1}{2}C_2 = 1$$

$$p' = C_1\frac{\sqrt{19}}{2} + \frac{38}{2} = 1$$

Solving:

$$C_1 = \frac{-36}{\sqrt{19}}$$

The final result is:

$$p = e^{-\frac{1}{2}t}\left[\frac{-36}{\sqrt{19}}\sin\left(\frac{\sqrt{19}}{2}t\right) - 38\cos\left(\frac{\sqrt{19}}{2}t\right)\right] + 50$$